10/27/11 - Categorical twists and split-generation

- We've built: given $V = L_0 \subset M$ exact Lag. sphere, L_1, $L_2 \simeq \tau_V(L_1)$ exact Lags.
  (+ grading data etc...)

$$\Rightarrow (\ast) \quad \begin{tikzcd} CF(L_0,L_1)\otimes L_0 \arrow[r, "t"', "\text{(tautological)}"] \arrow[rr, bend right, "k\ (\deg. -1)"'] & L_1 \arrow[r, "c"] & L_2 \end{tikzcd}$$

- twisted complex $\left(\text{ie. } \begin{array}{l}\mu^1(c)=0\\ \mu^2(c,t)\pm\mu^1(k)=0\end{array}\right)$

+ similarly for Yoneda modules

ie. $\forall Q \in \mathcal{F}(M)$, $$\begin{tikzcd} CF(L_0,L_1)\otimes CF(Q,L_0) \arrow[r, "\mu^2"] \arrow[rr, bend right, "\mu^2(k(\cdot),\cdot)"'] & CF(Q,L_1) \arrow[r, "\mu^2(c,\cdot)"] & CF(Q,L_2) \end{tikzcd}$$

+ naturality wrt Q

& <u>acyclic</u> (ie. corresponding Yoneda module has $H^* \simeq 0$).

- In $Tw\,\mathcal{F}(M)$, let $K =$ mapping cone of $\{CF(L_0,L_1)\otimes L_0 \xrightarrow{t} L_1\}$
$$= \Big(\big(CF(L_0,L_1)\otimes L_0\big)[1] \oplus L_1\ ,\ t\Big)$$

Then $K \xrightarrow{(k,c)} L_2$ is a <u>closed</u> hom. of twisted complexes

ie. $$\begin{tikzcd} CF(L_0,L_1)\otimes L_0 \arrow[d, "t"'] \arrow[r, "k"] & L_2 \\ \oplus\ L_1 \arrow[ur, "c"'] & \end{tikzcd} \qquad \mu^1_{Tw}((k,c)) = \mu^2(c,t) \pm \mu^1(k) = 0$$

and (k,c) is a <u>quasi isom.</u> (since cohomology of $(\ast) \simeq$ cohom. of $0 \to H(K) \xrightarrow[H(k,c)]{} H(L_2) \to 0$)

Hence $K \underset{(k,c)}{\xrightarrow{\sim}} L_2$.

<u>Def</u>: categorical <u>twist functor</u> $T_{L_0} : \overset{\text{mod-}\mathcal{F}(M)}{Tw\,\mathcal{F}(M)} \to \overset{\text{mod-}\mathcal{F}(M)}{Tw\,\mathcal{F}(M)}$ (or any triangulated enlargement)

$$T_{L_0}(L_1) := \mathrm{Cone}\big(\mathrm{Hom}(L_0,L_1)\otimes L_0 \xrightarrow[\text{tautological}]{t} L_1\big)$$

(& natural def. on morphisms)

<u>We've proved</u>: $\tau_{L_0}(L_1) \simeq T_{L_0}(L_1) \quad \forall L_1$

(and in fact this is natural wrt L_1, ie. the functors τ_{L_0} and T_{L_0} are quasi isomorphic)

Now: $K = T_{L_0}(L_1) = \{ CF(L_0,L_1) \otimes L_0 \xrightarrow{t} L_1 \}$ sits in an exact triangle

$$\begin{tikzcd} CF(L_0,L_1)\otimes L_0 \arrow[rr, "t"] & & L_1 \arrow[dl, "i \text{ incl.}"] \\ & K \arrow[ul, "\pi \text{ proj.}"] & \end{tikzcd}$$

but $K \xrightarrow[(k,c)]{\simeq} L_2$ quasi iso. (invertible up to homotopy)

$\Rightarrow$ by def$^{\underline{n}}$ of exact triangles, can replace K by quasiisomorphic L_2 & get exact triangle

$$\begin{tikzcd} CF(L_0,L_1)\otimes L_0 \arrow[rr, "t"] & & L_1 \arrow[dl, "\mu^2_{Tw}((k,c),\, i) = c"] \\ & L_2 \arrow[ul, "\substack{\text{(obtained from } \pi \\ \text{and quasi inverse} \\ \text{to } (k,c))}"] & \end{tikzcd}$$

recall: $\mu^2_{Tw}\left(\begin{tikzcd} & L_1 \arrow[d, "i"] \\ CF(L_0,L_1)\otimes L_0 \arrow[r, bend left, "t"] \arrow[dr, "k"'] & \oplus L_1 \arrow[d, "c"] \\ & L_2 \end{tikzcd},\ i \right)$

<u>Application:</u> assume Y smooth projective variety,

$X = X_0 \subseteq Y$ smooth alg. hypersurface $= \{ s_0 = 0 \}$, $s_0 \in H^0(Y, K_Y^{-1})$,

(implies $c_1(X) = 0$ by adjunction),

X fiber of a <u>Lefschetz pencil</u>: ie.

$\exists$ other section $s_\infty \in H^0(Y, K_Y^{-1})$, $X_\infty = \{ s_\infty = 0 \}$ smooth, $X_0 \pitchfork X_\infty$,

& s.t. $\pi = s_0 / s_\infty : E = Y \setminus X_\infty \longrightarrow \mathbb{C}$ is a Lefschetz fibration,

w/ fiber $M = \pi^{-1}(0) \simeq X_0 \setminus (X_0 \cap X_\infty)$

<u>Ex:</u> $Y = \mathbb{CP}^{n+1}$, s_0, s_∞ homogeneous polynomials of degree $n+2$

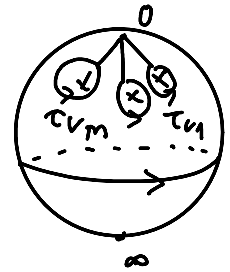
($X =$ genus 1 cubic curve $\subset \mathbb{CP}^2$
quartic K3 surface $\subset \mathbb{CP}^3$
...)

Choose a basis of vanishing paths for π

Let $V_1, \dots, V_m \subset M$ be the vanishing cycles

<u>Thm</u> (Seidel) ‖ $V_1, \dots, V_m$ <u>split-generate</u> $\mathcal{F}(M)$, ie. any object of $\mathcal{F}(M)$ is $\simeq$ in $Tw\, \mathcal{F}(M)$ to a direct summand in a twisted complex built out of copies of $V_1, \dots V_m$.

<u>Key observation:</u> monodromy at ∞ of L. fibration π: classically $\simeq id$ because can extend π to L. fibration over $\mathbb{CP}^1$ by adding smooth fiber $X_\infty \setminus (X_0 \cap X_\infty)$

So: 

$$\varphi := \tau_{V_1} \circ \dots \circ \tau_{V_m} \simeq id \quad \text{in } Symp(M)$$

however, Seidel shows:

as <u>graded</u> symplectomorphisms, $\varphi := \tau_{V_1} \circ \dots \circ \tau_{V_m} \simeq id[2]$

so $\forall L \in \mathcal{F}(M)$, $\tau_{V_1} \circ \dots \circ \tau_{V_m}(L) \simeq L[2]$

[this is due to monodromy rel. $\partial(\text{fiber}) \simeq S^1$-bundle over $(V_0 \cap X_\infty)$ actually being a boundary twist]

Now: $\tau_{V_1} \circ \dots \circ \tau_{V_m}(L) \simeq T_{V_1} \circ \dots \circ T_{V_m}(L)$ = iterated mapping cone:

$$T_{V_m}(L) = [\, CF(V_m, L) \otimes V_m \longrightarrow L \,]$$

$$T_{V_{m-1}}(T_{V_m}(L)) = \left[\begin{tikzcd} CF(V_m,L)\otimes CF(V_{m-1},V_m)\otimes V_{m-1} \arrow[r] \arrow[d] \arrow[dr] & CF(V_{m-1},L)\otimes V_{m-1} \arrow[d] \\ CF(V_m,L)\otimes V_m \arrow[r] & L \end{tikzcd} \right]$$

(all maps tautological)

...

$$(**)\ T_{V_1} \circ \dots \circ T_{V_m}(L) = \left(\bigoplus_{i_0 < \dots < i_k} CF(V_{i_k}, L) \otimes \underset{\circlearrowright}{CF(V_{i_{k-1}}, V_{i_k})} \otimes \dots \otimes CF(V_{i_0}, V_{i_1}) \otimes V_{i_0} \right) \oplus L$$

where maps are all tautologically defined by A_∞-operations in $\mathcal{F}(M)$

(Yoneda module: $CF(Q, T_{V_1} \circ \dots \circ T_{V_m}(L)) = \left(\bigoplus CF(\dots, L) \otimes \dots \otimes CF(Q, \dots) \right)$

δ = all A_∞-maps (bar complex))

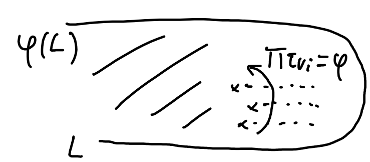
Rewrite this as: $T_{V_1} \circ \dots \circ T_{V_m}(L) = Cone(K \xrightarrow[f]{} L)$

where K = twisted complex involving copies of $V_1 \dots V_m$ (= all but L in $(**)$).

Thus: we have a quasi-isom. $\varphi(L) \simeq Cone(K \xrightarrow[f]{} L)$ in $Tw\, \mathcal{F}(M)$,

hence an exact triangle

$$\begin{tikzcd} K \arrow[rr, "f"] & & L \arrow[dl, "c"] \\ & \varphi(L) \arrow[ul, "{[1]}"] & \end{tikzcd}$$



where actually $c \in CF(L, \varphi(L))$ counts pseudoholom. sections of our L. fibration

Can often show $c = 0$ by dimension argument. Here:
iterate further, consider $\varphi^k = (\tau_{V_1} \cdots \tau_{V_m})^k$ where $k > \frac{n}{2}$.

Then we have similarly

complex built out of $V_1 \ldots V_m$

$$\begin{tikzcd} \tilde{K} \arrow[r, "\tilde{f}"] & L \arrow[dl, "\tilde{c}"] \\ \varphi^k(L) \simeq L[2k] \arrow[u] & \end{tikzcd}$$

however $[\tilde{c}] \in HF^0(L, \varphi^k(L)) \cong HF^{2k}(L,L) = 0$ for degree reasons!

So $\tilde{c} \sim 0$, and in fact $\tilde{K}$ is quasi isom. to $L \oplus L[2k]$. QED.

In general, don't actually need monodromy = id[shift] if can ensure that the map c vanishes $\forall L$
(ie. the natural transformation $id \to \varphi$ given by counting sections of our L. fibration is nullhomotopic).